On the blow-up threshold for weakly coupled nonlinear Schrödinger equations

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Abstract

We study the Cauchy problem for a system of two coupled nonlinear focusing Schrödinger equations arising in nonlinear optics. We discuss when the solutions are global in time or blow-up in finite time. Some results, in dependence of the data of the problem, are proved; in particular we prove, for suitable values of the parameters, that the blow-up threshold (if the nonlinearity has the critical growth) is a universal constant.

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1. Introduction

In this paper, we consider the following Cauchy problem for two coupled nonlinear Schrödinger equations:

$$\begin{cases} i\phi_t + \Delta\phi + (|\phi|^{2p} + \beta|\psi|^{p+1}|\phi|^{p-1})\phi = 0 \\ i\psi_t + \Delta\psi + (|\psi|^{2p} + \beta|\phi|^{p+1}|\psi|^{p-1})\psi = 0 \\ \phi(0,x) = \phi_0(x) \qquad \psi(0,x) = \psi_0(x), \end{cases} \quad (1)$$

where $\phi, \psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}, \phi_0, \psi_0 : \mathbb{R}^n \to \mathbb{C}, p \geqslant 0$ and β is a real positive constant.

This kind of problems arises as a model for propagation of polarized laser beams in birefringent Kerr medium in nonlinear optics (see, for example, [3, 8, 15, 18, 26, 28, 29] and the references therein for a complete discussion of the physics of the problem). The two functions ϕ and ψ are the components of the slowly varying envelope of the electrical field, t is the distance in the direction of propagation, x are the orthogonal variables and Δ is the diffraction operator. The case $n = 1$ corresponds to propagation in a planar geometry, $n = 2$

is the propagation in a bulk medium and $n = 3$ is the propagation of pulses in a bulk medium with time dispersion (in this case x includes also the time variable).

The focusing nonlinear terms in (1) describe the dependence of the refraction index of the material on the electric field intensity and the birefringence effects. The parameter $\beta > 0$ has to be interpreted as the *birefringence* intensity and describes the coupling between the two components of the electric-field envelope. The case $p = 1$ (i.e. cubic nonlinearities in (1)) is known as Kerr nonlinearity in the physical literature.

We are interested in a slightly more general model, in order to cover the physical cases and to discuss some results about Cauchy problem (1) from a more general point of view.

The aim of this paper is to study the $H^1 \times H^1$ well-posedness of problem (1), with respect to the nonlinearity, in analogy with the case of the single focusing nonlinear Schrödinger equation

$$\begin{cases} \mathrm{i}\psi_t + \Delta\psi + |\psi|^{2p}\psi = 0 \\ \psi(0, x) = f(x), \end{cases} \tag{2}$$

for $\psi : \mathbb{R}^{1+n} \to \mathbb{C}$ and $f : \mathbb{R}^n \to \mathbb{C}$.

The study of the single nonlinear Schrödinger equation begins with the pioneering works [26, 28], where the collapse of waves (blow-up of solutions, in the sequel) is deeply analysed by an experimental and theoretical point of view. In the last 30 years many authors worked on this equation, in order to make clearness on the mathematical properties of (2); the well-known results can be summarized as follows. By standard scaling arguments it is possible to claim that the critical exponent for the H^1 local well-posedness of (2) is $p = 2/(n-2)$ (see [6, 24]). Indeed, contraction techniques based on Strichartz estimates (see [10, 14]) permit to prove that (2) is locally well-posed in H^1 for $p < 2/(n-2)$ (see [9, 6, 24]). To pass from local to global well-posedness, it is natural to introduce the energy function given by

$$E(t) = \frac{1}{2}\|\nabla\psi\|_2^2 - \frac{1}{2p+2}\|\psi\|_{2p+2}^{2p+2},$$

that is conserved along any solution ψ of (2). For $p < 2/n$ the unique local H^1 solution can be extended globally in time by a continuation argument. In the *critical case* $p = 2/n$, we can also extend local solutions to global ones, provided the initial data are not too large in the L^2. Finally, for $2/n \leqslant p < 2/(n-2)$ without restriction on the data, it is possible to prove that the L^2 norm of the gradient, in general, blows up in a finite time (see, e.g., the original work [11] or [6, 24]).

By a physical point of view it is very interesting to determine the threshold for the initial mass of the wave packet, that is the L^2 norm of the initial datum, which separates global existence and blow-up in the critical case. We recall that $\psi = \mathrm{e}^{\mathrm{i}t}u(x) \in H^1$ is a *ground-state solution* for (2) if u is a nonzero critical point of the action functional

$$A(u) = E(u) + \frac{1}{2}\|u\|_2^2 = \frac{1}{2}\left(\|\nabla u\|_2^2 + \|u\|_2^2\right) - \frac{1}{2+4/n}\|u\|_{2+4/n}^{2+4/n},$$

having the smallest action level; clearly, u solves

$$-\Delta u + u = |u|^{4/n}u. \tag{3}$$

In [25], Weinstein proved that if the initial mass is smaller than a constant C_n, depending only on the space dimension n, than there exists a unique global H^1 solution; moreover, C_n is the L^2 norm of any ground-state solutions of (2) and can be numerically estimated. Moreover, we want to point out that this kind of phenomena for the single equation present other kind of universality properties related, for example, to the blow-up profile (see [3, 19, 20] and the references therein).

Our main goal is to state the analogous result for the coupled system (1). The critical exponent for the local $H^1 \times H^1$ well-posedness has to be again $p = 2/(n-2)$; so for $p < 2/(n-2)$ it is possible to prove that (1) possesses a unique local solution (see remark 4.2.13 in [6] and section 2). The natural energy for (1) is the following:

$$E(t) = \frac{1}{2}\left(\|\nabla\phi\|_2^2 + \|\nabla\psi\|_2^2\right) - \frac{1}{2p+2}\left(\|\phi\|_{2p+2}^{2p+2} + 2\beta\|\phi\psi\|_{p+1}^{p+1} + \|\psi\|_{2p+2}^{2p+2}\right). \tag{4}$$

Also here it is possible to prove that $E(t)$ is conserved (see section 2); hence the same techniques for the single equation can be applied to extend local solution to global ones. Now we can state our first result.

Theorem 1. *Assume that $p < 2/n$. Then the Cauchy problem (1) is globally well-posed in $H^1 \times H^1$, i.e. for any $(\phi_0, \psi_0) \in H^1 \times H^1$ there exists a unique solution $(\phi, \psi) \in C\left(\mathbb{R}; H^1 \times H^1\right)$.*

Also here ground-state solutions of (1) play a crucial role in the dynamics of the system. In this case, they are solutions of the form $(\phi, \psi) = e^{it}(u(x), v(x))$, where the functions u and v have to be a least-action solution of a elliptic system (see (12)).

Since the birefringence tends to split a pulse into two pulses in two different polarization directions, the properties of the ground-state solutions of (1) depend strongly on the coupling parameter. If β is sufficiently small, that is the interaction is weak, any ground state is a *scalar* solution, i.e. one of the two components is zero. On the other hand, when the birefringence is strong, $\beta \gg 1$, we have *vector* ground states, i.e. all the components are distinct from zero (see [1, 17]). This suggests that the blow-up phenomena, in the critical case, should also depend on the parameter β. It is natural to claim that in a weak-interaction regime the behaviour has to be exactly the same of the single equation. Otherwise if $\beta \gg 1$, we expect that the analogous of the Weinstein threshold C_n should depend on β also. These claims are proved in the following main theorem.

Theorem 2. *Assume that $p = 2/n$. Then there exists a constant $C = C_{n,\beta}$ such that the Cauchy problem (1) is globally well-posed in $H^1 \times H^1$ if*

$$\|\phi_0\|_2^2 + \|\psi_0\|_2^2 < C.$$

Moreover, there exists a pair (ϕ_0, ψ_0) such that $\|\phi_0\|_2^2 + \|\psi_0\|_2^2 = C_{n,\beta}$ and the corresponding solution blows up in a finite time. The constant $C_{n,\beta}$ has the following behaviour:

$$\begin{cases} C_{n,\beta} = C_n & \text{if} \quad \beta \leqslant 2^{2/n} - 1, \\ C_{n,\beta} \geqslant C_n \dfrac{(1+\beta)}{2^{2/n}} & \text{if} \quad \beta \geqslant 2^{2/n} - 1, \end{cases} \tag{5}$$

where C_n is the blow-up threshold of a single equation.

Remark 3. In the supercritical case the solution of the Cauchy problem for (1) exists locally in time, by the results in [6]. It is possible to prove that the solution exists globally in time if the assumption $\|\phi_0\|^2, \|\psi_0\|^2 \ll 1$ is satisfied (see theorem 6.1.1 in [6]).

Remark 4. As observed above, the Kerr nonlinearities (corresponding to $p = 1$) are physically relevant; in this case the system (1) becomes

$$\begin{cases} i\phi_t + \Delta\phi + (|\phi|^2 + \beta|\psi|^2)\phi = 0 \\ i\psi_t + \Delta\psi + (|\psi|^2 + \beta|\phi|^2)\psi = 0. \end{cases} \tag{6}$$

The above results (theorems 1 and 2) can be summarized in the following way:

(i) if $n = 1$ the Cauchy problem (6) is globally (in time) well-posed in $H^1 \times H^1$,

(ii) if $n = 2$ the cubic nonlinearity is critical, so the Cauchy problem (6) is globally well-posed for small data; moreover the blow-up threshold $C_{2,\beta}$ is constant for any $\beta \leqslant 1$ and tends to infinity as $\beta \to +\infty$,

(iii) if $n \geqslant 3$ a solution of the Cauchy problem (6) exists globally in time provided the initial datum is sufficiently small in $L^2 \times L^2$.

The single equation with a Kerr nonlinearity has been studied also in bounded domains or on compact manifolds (see, for example, [4, 5] and the references therein). The study of coupled nonlinear Schrödinger equations in bounded domains or on compact manifolds should be interesting in view to extend the results for a single equation.

The paper is organized in the following way. Section 2 is devoted to the proofs of the existence results above, in section 3 it is proved a Gagliardo–Nirenberg inequality (see (9)) which is the fundamental tool to obtain theorems 1 and 2. Section 4 deals with a blow-up result which shows the sharpness of constant $C_{n,\beta}$, while in section 5 the proof of theorem 2 is completed.

2. Global existence results

The first part of this work is devoted to the proof of theorem 1. The theory for the single nonlinear Schrödinger equation (2) was developed in [9, 13]; the proof of the local well-posedness is a contraction argument based on Strichartz estimates, and the conservation of both the mass and the energy allows us to extend the local solution globally in time. The fixed point technique also works in the case of a system, hence problem (1) is locally well-posed in H^1 for $0 \leqslant p \leqslant 2/(n-2)$. We omit here the straightforward computations (see, for example, remarks 4.2.13 and 4.3.4 in [6]).

Let us now study the conservation laws for system (1). Multiplying the equations in (1) by $\overline{\phi}$ and $\overline{\psi}$, respectively, integrating in x and taking the resulting imaginary parts, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\phi\|_2^2 = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t} \|\psi\|_2^2 = 0, \tag{7}$$

i.e. the conservation of the masses. Note that these computations make sense if ϕ, ψ are H^1 solutions (it is possible to prove (7) also in the case of L^2 solutions, following for example the techniques of [21]).

Now we consider the energy $E(t)$ defined in (4). Let (ϕ, ψ) be a solution to (1); multiplying the equations in (1) by $\overline{\phi}_t$ and $\overline{\psi}_t$, respectively, integrating by parts in x and taking the resulting real parts, we easily obtain the energy conservation

$$E'(t) = 0. \tag{8}$$

This formal computation needs H^2 regularity for ϕ, ψ, but (8) makes sense (and can be proved) also for H^1 solutions. To prove this, following exactly the same computations of Ozawa in [21], proposition 2, we omit here the details.

In order to obtain an *a priori* control on the gradient of the solutions, we introduce a Gagliardo–Nirenberg inequality (see section 3):

$$\left(\|u\|_{2p+2}^{2p+2} + 2\beta \|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2} \right) \leqslant C_{n,p,\beta} \left(\|u\|_2^2 + \|v\|_2^2 \right)^{p+1-p\frac{n}{2}} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right)^{p\frac{n}{2}}, \tag{9}$$

that gives the following bound from below:

$$E(t) \geqslant \frac{1}{2} \left(\|\nabla \phi\|_2^2 + \|\nabla \psi\|_2^2 \right) \left[1 - \frac{C_{n,p,\beta}}{p+1} \left(\|\phi\|_2^2 + \|\psi\|_2^2 \right)^{p+1-p\frac{n}{2}} \left(\|\nabla \phi\|_2^2 + \|\nabla \psi\|_2^2 \right)^{p\frac{n}{2}-1} \right]. \tag{10}$$

If $p < 2/n$, we easily see by (10) that the norms $\|\nabla\phi\|_2$, $\|\nabla\psi\|_2$ cannot blow up in a finite time, because of the conservation of both the mass and the energy; as a consequence, global well-posedness in H^1 is proved in the subcritical range. The power $p = 2/n$ is critical, in the sense that this nonlinearity is sufficiently high to generate H^1 solutions blowing up in a finite time. On the other hand, also in this case, the smallness assumption

$$\left(\|\phi\|_2^2 + \|\psi\|_2^2\right)^{2/n} < \frac{p+1}{C_{n,p,\beta}} \tag{11}$$

allows by (10) to obtain the same *a priori* control for the gradient in terms of the energy, hence the global existence in theorems 1 and 2 is proved. The last part of theorem 2 is proved in section 4.

3. Gagliardo–Nirenberg inequality

Our next step is to discuss, following the approach of [25], the behaviour of the best constant $C_{n,p,\beta}$ in the Gagliardo–Nirenberg inequality (9); this will allow us to understand which is the critical initial level defining the border line between global well-posedness and blow-up phenomena. This involves the existence of minimal energy stationary solutions of (1) and allows us to clarify the concept of *ground state*.

Consider the functional

$$J_{n,p,\beta}(u, v) = \frac{\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{pn/2} \left(\|u\|_2^2 + \|v\|_2^2\right)^{p+1-pn/2}}{\left(\|u\|_{2p+2}^{2p+2} + 2\beta\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}\right)}, \qquad u, v \in H^1;$$

the infimum of $J_{n,p,\beta}$ on $H^1 \times H^1$ is clearly the reciprocal of the best constant $C_{n,p,\beta}$ in (9). First of all we want to point out that, for any $u, v \in H^1$ and for any $\mu, \lambda > 0$, if we set $u_{\mu,\lambda}(x) = \mu u(\lambda x)$ and $v_{\mu,\lambda}(x) = \mu v(\lambda x)$ it follows

$$\begin{aligned}
\|u_{\mu,\lambda}\|_2^2 &= \mu^2\lambda^{-n}\|u\|_2^2, & \|\nabla u_{\mu,\lambda}\|_2^2 &= \mu^2\lambda^{2-n}\|\nabla u\|_2^2, \\
\|v_{\mu,\lambda}\|_2^2 &= \mu^2\lambda^{-n}\|v\|_2^2, & \|\nabla v_{\mu,\lambda}\|_2^2 &= \mu^2\lambda^{2-n}\|\nabla v\|_2^2, \\
\|u_{\mu,\lambda}\|_{2p+2}^{2p+2} &= \mu^{2p+2}\lambda^{-n}\|u\|_{2p+2}^{2p+2}, & \|v_{\mu,\lambda}\|_{2p+2}^{2p+2} &= \mu^{2p+2}\lambda^{-n}\|v\|_{2p+2}^{2p+2},
\end{aligned}$$

so that

$$J_{n,p,\beta}(u_{\mu,\lambda}, v_{\mu,\lambda}) = J_{n,p,\beta}(u, v).$$

Assume that the infimum of $J_{n,p,\beta}$ is achieved by $(\tilde{u}, \tilde{v})$: since the value of the functional is invariant with respect to the above scalings, we can assume that the best constant in (9) is achieved by the pair $(\tilde{u}, \tilde{v})$ such that

$$\left(\|\tilde{u}\|_2^2 + \|\tilde{v}\|_2^2\right) = \left(\|\nabla\tilde{u}\|_2^2 + \|\nabla\tilde{v}\|_2^2\right) = 1.$$

Therefore, $(\tilde{u}, \tilde{v})$ is a weak solution of the following system of two weakly coupled elliptic equations:

$$\begin{cases}
-\dfrac{pn}{2}\Delta\tilde{u} + \dfrac{(2-n)p+2}{2}\tilde{u} = \dfrac{1}{C_{n,p,\beta}}(|\tilde{u}|^{2p} + \beta|\tilde{u}|^{p-1}|\tilde{v}|^{p+1})\tilde{u} \\[2mm]
-\dfrac{pn}{2}\Delta\tilde{v} + \dfrac{(2-n)p+2}{2}\tilde{v} = \dfrac{1}{C_{n,p,\beta}}(|\tilde{v}|^{2p} + \beta|\tilde{v}|^{p-1}|\tilde{u}|^{p+1})\tilde{v}.
\end{cases}$$

Now consider the pair $(\tilde{u}_{\mu,\lambda}, \tilde{v}_{\mu,\lambda})$, corresponding to the choice of parameters

$$\mu = \left(\frac{2}{C_{n,p,\beta}(2p+2-pn)}\right)^{1/2p}, \qquad \lambda = \left(\frac{pn}{(2p+2-pn)}\right)^{1/2};$$

this pair solves the following elliptic system:

$$\begin{cases} -\Delta \tilde{u}_{\mu,\lambda} + \tilde{u}_{\mu,\lambda} = (|\tilde{u}_{\mu,\lambda}|^{2p} + \beta|\tilde{u}_{\mu,\lambda}|^{p-1}|\tilde{v}_{\mu,\lambda}|^{p+1})\tilde{u}_{\mu,\lambda} \\ -\Delta \tilde{v}_{\mu,\lambda} + \tilde{v}_{\mu,\lambda} = (|\tilde{v}_{\mu,\lambda}|^{2p} + \beta|\tilde{v}_{\mu,\lambda}|^{p-1}|\tilde{u}_{\mu,\lambda}|^{p+1})\tilde{v}_{\mu,\lambda}. \end{cases} \tag{12}$$

Note that the preceding system is variational in nature, so that any (weak) solution is a critical point of the functional

$$I_{n,p,\beta}(u,v) = \frac{1}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u\|_2^2 + \|v\|_2^2\right)$$
$$- \frac{1}{2p+2}\left(\|u\|_{2p+2}^{2p+2} + \beta\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}\right).$$

Recently, the problem of existence of positive solutions for elliptic systems of this kind has been studied by many authors (see, for example, [1, 2, 7, 16, 17, 23, 27]). In [17], particular attention is given to the existence and some qualitative properties of the ground-state solutions of (12): a ground-state solution is a nontrivial solution (i.e. distinct from the pair $(0,0)$) which has the least critical level. In particular, it is possible to prove the existence of ground-state solutions for system (12) solving the following minimization problem:

$$\inf_{(u,v)\in\mathcal{N}} I_{n,p,\beta}(u,v), \tag{13}$$

where $\mathcal{N} \subset H^1 \times H^1$ is the Nehari manifold, that is

$$\mathcal{N} = \left\{(u,v) \neq (0,0) : \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u\|_2^2 + \|v\|_2^2 = \|u\|_{2p+2}^{2p+2} + 2\beta\|uv\|_{p+1}^{p+1}\|v\|_{2p+2}^{2p+2}\right\}.$$

Since $\mathcal{N}$ is a smooth (of class C^2) manifold containing all the nontrivial critical points of the functional, that is all the weak solutions of (12), clearly a ground-state solution has to realize the minimum. We want to point out that $I_{n,p,\beta}$ is bounded from below on $\mathcal{N}$, so the minimization problem (13) is well-posed; moreover, it is possible to prove that any minimizing sequences is compact (up to translations) and that the minimum is achieved.

Let

$$m_{n,p,\beta} = \inf_{\mathcal{N}} I_{n,p,\beta} = I_{n,p,\beta}(\tilde{u}_{\mu,\lambda}, \tilde{v}_{\mu,\lambda})$$

be the level of any ground-state solution of (12); we want to prove that there is a direct relation between $C_{n,p,\beta}$ and $m_{n,p,\beta}$. Recalling that any critical point of $I_{n,p,\beta}$ is a weak solution of (12), multiplying (12) by $(\tilde{u}_{\mu,\lambda}, \tilde{v}_{\mu,\lambda})$ and integrating on $\mathbb{R}^n$ we obtain that

$$\|\nabla\tilde{u}_{\mu,\lambda}\|_2^2 + \|\tilde{u}_{\mu,\lambda}\|_2^2 = \|\tilde{u}_{\mu,\lambda}\|_{2p+2}^{2p+2} + \beta\|\tilde{u}_{\mu,\lambda}\tilde{v}_{\mu,\lambda}\|_{p+1}^{p+1},$$
$$\|\nabla\tilde{v}_{\mu,\lambda}\|_2^2 + \|\tilde{v}_{\mu,\lambda}\|_2^2 = \|\tilde{v}_{\mu,\lambda}\|_{2p+2}^{2p+2} + \beta\|\tilde{u}_{\mu,\lambda}\tilde{v}_{\mu,\lambda}\|_{p+1}^{p+1}. \tag{14}$$

Moreover in this case, Pohozaev identity reads

$$\frac{n-2}{2}\left(\|\nabla\tilde{u}_{\mu,\lambda}\|_2^2 + \|\nabla\tilde{v}_{\mu,\lambda}\|_2^2\right) + \frac{n}{2}\left(\|\tilde{u}_{\mu,\lambda}\|_2^2 + \|\tilde{v}_{\mu,\lambda}\|_2^2\right)$$
$$= \frac{n}{2p+2}\left(\|\tilde{u}_{\mu,\lambda}\|_{2p+2}^{2p+2} + 2\beta\|\tilde{u}_{\mu,\lambda}\tilde{v}_{\mu,\lambda}\|_{p+1}^{p+1} + \|\tilde{v}_{\mu,\lambda}\|_{2p+2}^{2p+2}\right). \tag{15}$$

Putting together the above identities we have that

$$\mu^2\lambda^{2-n}\left(\|\nabla\tilde{u}\|_2^2 + \|\nabla\tilde{v}\|_2^2\right) = \left(\|\nabla\tilde{u}_{\mu,\lambda}\|_2^2 + \|\nabla\tilde{v}_{\mu,\lambda}\|_2^2\right) = nm_{n,p,\beta},$$

$$\mu^2\lambda^{-n}\left(\|\tilde{u}\|_2^2 + \|\tilde{v}\|_2^2\right) = \left(\|\tilde{u}_{\mu,\lambda}\|_2^2 + \|\tilde{v}_{\mu,\lambda}\|_2^2\right) = \left(2 - n + \frac{2}{p}\right)m_{n,p,\beta},$$

$$\mu^{2p+2}\lambda^{-n}\left(\|\tilde{u}\|_{2p+2}^{2p+2} + 2\beta\|\tilde{u}\tilde{v}\|_{p+1}^{p+1} + \|\tilde{v}\|_{2p+2}^{2p+2}\right)$$
$$= \left(\|\tilde{u}_{\mu,\lambda}\|_{2p+2}^{2p+2} + 2\beta\|\tilde{u}_{\mu,\lambda}\tilde{v}_{\mu,\lambda}\|_{p+1}^{p+1} + \|\tilde{v}_{\mu,\lambda}\|_{2p+2}^{2p+2}\right) = \frac{2p+2}{p}m_{n,p,\beta}.$$

All the above calculations imply that the following equalities hold:

$$\frac{1}{C_{n,p,\beta}} = J_{n,p,\beta}(\tilde{u}, \tilde{v}) = J_{n,p,\beta}(\tilde{u}_{\mu,\lambda}, \tilde{v}_{\mu,\lambda}) = m^p_{n,p,\beta} \frac{n^{pn/2}(2p+2-pn)^{p+1-pn/2}}{2(p+1)p^{p-pn/2}}. \qquad (16)$$

Note that, in the critical case $p = 2/n$, (16) becomes

$$\frac{1}{C_{n,2/n,\beta}} = 2^{2/n}\frac{n}{n+2}m^{2/n}_{n,2/n,\beta} = \frac{n}{n+2}\left(\|\tilde{u}_{\mu,\lambda}\|^2_2 + \|\tilde{v}_{\mu,\lambda}\|^2_2\right)^{2/n}. \qquad (17)$$

The arguments above, in particular (16), show that a minimum point of $J_{n,p,\beta}$, through a suitable scaling, has to correspond to a ground-state solution of (12) (or to a least energy nontrivial critical point of $I_{n,p,\beta}$). Now, since in [17] it is proved the existence of ground-state solutions to (12), we have obtained the existence of a minimum point for the functional $J_{n,p,\beta}$; this shows that inequality (9) is sharp and that there exists at least a pair of functions for which equality holds. More generally, we have proved that the functionals $J_{n,p,\beta}$ and $I_{n,p,\beta}$ possess the same number of critical values.

The validity of inequality (9) follows by the above arguments.

4. Blow-up results

In view to prove the sharpness of the constant C in the statement of theorem 2, we introduce (following [11, 22]) another physically relevant quantity, that plays a crucial role in the analysis of blow-up phenomena: the *variance* $V(t)$, which is defined by

$$V(t) = \int |x|^2|\phi(t,x)|^2\,dx + \int |x|^2|\psi(t,x)|^2\,dx. \qquad (18)$$

As in the case of a single Schrödinger equation, we will prove a relation between the time behaviour of V and that of the H^1 norm of the solutions: as we will see in the following, the precise calculation of the first and second derivatives of V in terms of the solutions of (1) is the main tool for the description of the blow-up (see, for example, [6] for a proof in the case of a single equation).

More precisely, we prove the following Lemma:

Lemma 5. *Let* (ϕ, ψ) *be a solution of system* (1) *on an interval* $I = (-t_1, t_1)$; *then, for each* $t \in I$, *the variance satisfies the following identities:*

$$V'(t) = 4\,\mathrm{Im}\int\left[(x\cdot\nabla\phi)\overline{\phi} + (x\cdot\nabla\psi)\overline{\psi}\right]dx, \qquad (19)$$

$$V''(t) = 8\int\left(|\nabla\phi|^2 + |\nabla\psi|^2\right)dx - \frac{4np}{p+1}\int\left(|\phi|^{2p+2} + 2\beta|\phi\psi|^{p+1} + |\psi|^{2p+2}\right)dx. \qquad (20)$$

Proof. We introduce the following notations:

$$z = (z^1, \ldots, z^n) \in \mathbb{C}^n;$$
$$z\cdot w = \sum_i z^i w^i, \qquad z, w \in \mathbb{C}^n;$$
$$u_i = \frac{\partial u}{\partial x_i}, \qquad u : \mathbb{R}^n \to \mathbb{C}.$$

Multiplying the equations in (1) by $2\overline{\phi}$ and $2\overline{\psi}$, respectively, and taking the resulting imaginary parts, we obtain

$$\frac{\partial}{\partial t}|\phi|^2 = -2\,\mathrm{Im}(\overline{\phi}\Delta\phi) = -2\nabla\cdot(\mathrm{Im}\,\overline{\phi}\nabla\phi), \tag{21}$$

$$\frac{\partial}{\partial t}|\psi|^2 = -2\,\mathrm{Im}(\overline{\psi}\Delta\psi) = -2\nabla\cdot(\mathrm{Im}\,\overline{\psi}\nabla\psi). \tag{22}$$

Now, multiplying (21) and (22) by $|x|^2$, and integrating by parts in x, we immediately obtain (19).

In order to prove (20), let us multiply the equations in (1) by $2(x\cdot\nabla\overline{\phi})$ and $2(x\cdot\nabla\overline{\psi})$, respectively, let us integrate in x and sum the equations for the real parts to get

$$0 = 2\,\mathrm{Re}\int \mathrm{i}[(x\cdot\nabla\overline{\phi})\phi_t(x\cdot\nabla\overline{\psi})\psi_t]\,\mathrm{d}x + 2\,\mathrm{Re}\int [(x\cdot\nabla\overline{\phi})\Delta\phi + (x\cdot\nabla\overline{\psi})\Delta\psi]\,\mathrm{d}x$$

$$+ 2\,\mathrm{Re}\int [(x\cdot\nabla\overline{\phi})(|\phi|^{2p} + \beta|\psi|^{p+1}|\phi|^{p-1})\phi + (x\cdot\nabla\overline{\psi})(|\psi|^{2p} + \beta|\phi|^{p+1}|\psi|^{p-1})\psi]\,\mathrm{d}x.$$

We rewrite the last identity in the form

$$\mathrm{I} = \mathrm{II} + \mathrm{III}, \tag{23}$$

where

$$\mathrm{I} = 2\,\mathrm{Re}\int \mathrm{i}[(x\cdot\nabla\overline{\phi})\phi_t(x\cdot\nabla\overline{\psi})\psi_t]\,\mathrm{d}x,$$

$$\mathrm{II} = -2\,\mathrm{Re}\int [(x\cdot\nabla\overline{\phi})\Delta\phi + (x\cdot\nabla\overline{\psi})\Delta\psi]\,\mathrm{d}x,$$

$$\mathrm{III} = -2\,\mathrm{Re}\int [(x\cdot\nabla\overline{\phi})(|\phi|^{2p} + \beta|\psi|^{p+1}|\phi|^{p-1})\phi$$

$$+ (x\cdot\nabla\overline{\psi})(|\psi|^{2p} + \beta|\phi|^{p+1}|\psi|^{p-1})\psi]\,\mathrm{d}x.$$

For the first term, we have

$$\mathrm{I} = -\mathrm{Re}\int \mathrm{i}\sum_j (x^j\overline{\phi}_j\phi_t - x^j\phi_j\overline{\phi}_t + x^j\overline{\psi}_j\psi_t - x^j\psi_j\overline{\psi}_t)\,\mathrm{d}x,$$

which can be written in the form

$$\mathrm{I} = \mathrm{Re}\int \mathrm{i}\sum_j x^j[(\overline{\phi}_j\phi)_t - (\phi\overline{\phi}_t)_j + (\overline{\psi}_j\psi)_t - (\psi\overline{\psi}_t)_j]\,\mathrm{d}x$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Re}\int \mathrm{i}[(x\cdot\nabla\overline{\phi})\phi + (x\cdot\nabla\overline{\psi})\psi]\,\mathrm{d}x + n\,\mathrm{Re}\int \mathrm{i}(\phi\overline{\phi}_t + \psi\overline{\psi}_t)\,\mathrm{d}x.$$

Now we evaluate the last equality using the equations in (1), obtaining

$$\mathrm{I} = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Im}\int [(x\cdot\nabla\phi)\overline{\phi} + (x\cdot\nabla\psi)\overline{\psi}]\,\mathrm{d}x - n\int (|\nabla\phi|^2 + |\nabla\psi|^2)\,\mathrm{d}x$$

$$+ n\int [(|\phi|^{2p} + \beta|\psi|^{p+1}|\phi|^{p-1})|\phi|^2 + (|\psi|^{2p} + \beta|\phi|^{p+1}|\psi|^{p-1})|\psi|^2]\,\mathrm{d}x$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Im}\int [(x\cdot\nabla\phi)\overline{\phi} + (x\cdot\nabla\psi)\overline{\psi}]\,\mathrm{d}x - n\int (|\nabla\phi|^2 + |\nabla\psi|^2)\,\mathrm{d}x$$

$$+ n\int (|\phi|^{2p+2} + 2\beta|\psi\phi|^{p+1} + |\psi|^{2p+2})\,\mathrm{d}x. \tag{24}$$

A multiple integration by parts in II gives the Pohozaev identity

$$\text{II} = (2 - n) \int (|\nabla\phi|^2 + |\nabla\psi|^2) \, dx. \tag{25}$$

As for the term III, we write it by components

$$\text{III} = -\sum_j \int \{x^j [|\phi|^{2p}(2\operatorname{Re}\overline{\phi}_j\phi) + |\psi|^{2p}(2\operatorname{Re}\overline{\psi}_j\psi)]$$

$$+ \beta x^j [|\phi|^{p-1}|\psi|^{p+1}(2\operatorname{Re}\overline{\phi}_j\phi) + |\psi|^{p-1}|\phi|^{p+1}(2\operatorname{Re}\overline{\psi}_j\psi)]\} \, dx. \tag{26}$$

Observe that

$$|\phi|^{2p}(2\operatorname{Re}\overline{\phi}_j\phi) + |\psi|^{2p}(2\operatorname{Re}\overline{\psi}_j\psi) = \frac{1}{p+1}\left(|\phi|_j^{2p+2} + |\psi|_j^{2p+2}\right),$$

$$|\phi|^{p-1}|\psi|^{p+1}(2\operatorname{Re}\overline{\phi}_j\phi) + |\psi|^{p-1}|\phi|^{p+1}(2\operatorname{Re}\overline{\psi}_j\psi) = \frac{2\beta}{p+1}(|\phi|^{p+1}|\psi|^{p+1})_j;$$

hence, integrating by parts in (26) we have

$$\text{III} = \frac{n}{p+1} \int (|\phi|^{2p+2} + |\psi|^{2p+2} + 2\beta|\phi|^{p+1}|\psi|^{p+1}) \, dx. \tag{27}$$

Finally, recollecting (23)–(25), (27) and (19), we complete the proof of (20). □

Remark 6. Note that (20) can be rewritten, recalling the definition of E, in the following equivalent form:

$$V''(t) = 16E(t) - 8\frac{np-2}{2p+2} \int (|\phi|^{2p+2} + 2\beta|\phi\psi|^{p+1} + |\psi|^{2p+2}) \, dx. \tag{28}$$

In the critical case $p = 2/n$ the equation above reduces to

$$V''(t) = 16E(t);$$

hence, the variance V of any solution of (1) with the negative initial energy vanishes in a finite time. For each $h : \mathbb{R}^n \to \mathbb{C}$ we can estimate

$$\|h\|_{L^2}^2 = \||h|^2\|_{L^1} \leqslant \||x|h\|_{L^2} \left\| \frac{h}{|x|} \right\|_{L^2}$$

by Cauchy–Schwartz inequality. As a consequence of the standard Hardy's inequality we obtain

$$\|h\|_{L^2}^2 \leqslant \||x|h\|_{L^2} \|\nabla h\|_{L^2}.$$

Applying the last inequality to any solution of (1) with the negative initial energy, since the mass is conserved and the variance vanishes in a finite time, the L^2 norm of the gradient needs necessarily to blow up in a finite time.

Remark 7. Consider the following pair:

$$\frac{e^{-i\frac{|x|^2-4}{4(1-t)}}}{(1-t)^{n/2}} \left(U\left(\frac{x}{1-t}\right), V\left(\frac{x}{1-t}\right) \right),$$

where (U, V) is a ground-state solution of (12). This is an explicit example of a blow-up solution, which shows that theorem 2 is sharp. Indeed the pair is a solution of (1) and has the initial value $e^{-i(|x|^2-4)/4}(U(x), V(x))$, which attains the critical blow-up threshold.

5. Conclusion: on the blow-up threshold

If $p = 2/n$, we have obtained the following characterization of the blow-up threshold:

$$\frac{1}{C_{n,2/n,\beta}} = \frac{1}{C_{n,\beta}} = \frac{n}{n+2} \left(\|U\|_2^2 + \|V\|_2^2 \right)^{2/n}, \tag{29}$$

where (U, V) is a ground-state solution of (12). In order to prove theorem 2 we have to estimate the quantities involved in (29).

In [17] (see theorem 2.5) it is proved that, if $\beta < 2^{2/n} - 1$, then any ground state of the elliptic system (12) is a scalar function, that is one of the components of the ground-state solution is zero. So we can assume, without loss of generality, that the ground state is $(z, 0)$, where $z \in H^1$ is the unique ground-state solution (see [25]) of the equation

$$-\Delta z + z = |z|^{\frac{4}{n}} z. \tag{30}$$

This implies that the constant $C_{n,\beta} = C_n$ depends only on n for any $\beta \leqslant 2^{2/n} - 1$, since the coupling parameter β now does not play a role in the problem of selecting the ground-state solution. Moreover, C_n is exactly the blow-up threshold for a single nonlinear Schrödinger equation, introduced and numerically computed in [25].

If $\beta \geqslant 2^{2/n} - 1$, $C_{n,\beta}$ depends on n and β and its expression is unknown, but we can estimate it using a suitable test pair. Let $\hat{z}$ be the unique positive ground-state solution of

$$-\Delta \hat{z} + \hat{z} = (1+\beta)|\hat{z}|^{\frac{4}{n}}\hat{z},$$

it is easy to see that the pair $(\hat{z}, \hat{z})$ is a positive solution of (12) for any β, and the following inequality holds

$$\frac{1}{C_{n,\beta}} = \frac{n}{n+2} \left(\|U\|_2^2 + \|V\|_2^2 \right)^{2/n} \leqslant \frac{n}{n+2} \left(2\|\hat{z}\|_2^2 \right)^{2/n}.$$

Clearly, we have an inequality since $(\hat{z}, \hat{z})$ could not be a ground-state solution of (12).

Using the scaling as in section 3 we can estimate $C_{n,\beta}$ with C_n. Recalling that z is the ground-state solution of (30) and noting that the L^2 norm of z is related to C_n (see (1.3) in [25] and also (17)) we obtain that

$$\|\hat{z}\|_{4/n}^2 = \frac{\|z\|_{4/n}^2}{1+\beta} = \frac{n+2}{n(1+\beta)C_n}.$$

Collecting the inequalities above we have

$$C_{n,\beta} \geqslant \left(\frac{1+\beta}{2^{2/n}} \right) C_n,$$

so that the claim is proved.

Now we give some concluding remarks:

Remark 8. We point out that the physical relevance of this result is the explicit bound for the constant $C_{n,\beta}$ given in (5), since it determines the range of initial data for which the collapse does not occur (see also [22, 25, 29]).

Equation (5) relies on the best constant in the Gagliardo–Nirenberg-type inequality (9), which is sharp. This leads to conjecture that the inequality in (5) is, in fact, an equality.

Remark 9. Note that it is possible to extend this argument to systems with more than two nonlinear Schrödinger equations, using some results about the elliptic counterpart contained in [1, 22].

Acknowledgment

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